



# Quantum Codes from Toric Surfaces

Johan P. Hansen

**Abstract**—A theory for constructing quantum error correcting codes from Toric surfaces by the Calderbank-Shor-Steane method is presented. In particular we study the method on toric Hirzebruch surfaces.

The results are obtained by constructing a dualizing differential form for the toric surface and by using the cohomology and the intersection theory of toric varieties.

**Index Terms**—Quantum computing, Codes, Block codes, Error correction codes.

## I. INTRODUCTION

In [1] and [2] the author developed methods to construct linear error correcting codes from toric varieties and derive the code parameters using the cohomology and the intersection theory on toric varieties. This method is generalized in section II to construct linear codes suitable for constructing quantum codes by the Calderbank-Shor-Steane method. Essential for the theory is the existence and the application of a dualizing differential form on the toric surface.

A.R. Calderbank [3], P.W. Shor [4] and A.M. Steane [5] produced stabilizer codes from linear codes containing their dual codes.

These two constructions are merged to obtain results for toric surfaces in section III-C. Similar merging has been done for algebraic curves with different methods by A. Ashikhmin, S. Litsyn and M.A. Tsfasman in [6].

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### A. Notation

- $\mathbb{F}_q$  – the finite field with  $q$  elements of characteristic  $p$ .
- $\mathbb{F}_q^*$  – the invertible elements in  $\mathbb{F}_q$ .
- $k = \overline{\mathbb{F}_q}$  – an algebraic closure of  $\mathbb{F}_q$ .
- $M \simeq \mathbb{Z}^2$  a free  $\mathbb{Z}$ -module of rank 2.
- $\square \subseteq M_{\mathbb{R}} = M \otimes_{\mathbb{Z}} \mathbb{R}$  – an integral convex polytope.
- $X = X_{\square}$  – the toric surface associated to the polytope  $\square$ .
- $T = T_N = U_0 \subseteq X$  – the torus.
- $S = |D_1| \cap |D_2| \subseteq X(\mathbb{F}_q)$  – the intersection of the supports of the divisors  $D_1$  and  $D_2$ .
- $\omega_X$  the sheaf of differential forms on  $X$ .

## II. THE METHOD OF TORIC VARIETIES

For the general theory of toric varieties we refer to [7], [8] and [9]. Here we will be using toric surfaces and we recollect some of their theory.

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### A. Toric surfaces and their cohomology

Let  $M$  be an integer lattice  $M \simeq \mathbb{Z}^2$ . Let  $N = \text{Hom}_{\mathbb{Z}}(M, \mathbb{Z})$  be the dual lattice with canonical  $\mathbb{Z}$  - bilinear pairing  $\langle \cdot, \cdot \rangle : M \times N \rightarrow \mathbb{Z}$ . Let  $M_{\mathbb{R}} = M \otimes_{\mathbb{Z}} \mathbb{R}$  and  $N_{\mathbb{R}} = N \otimes_{\mathbb{Z}} \mathbb{R}$  with canonical  $\mathbb{R}$  - bilinear pairing  $\langle \cdot, \cdot \rangle : M_{\mathbb{R}} \times N_{\mathbb{R}} \rightarrow \mathbb{R}$ .

Given a 2-dimensional integral convex polytope  $\square$  in  $M_{\mathbb{R}}$ . The support function  $h_{\square} : N_{\mathbb{R}} \rightarrow \mathbb{R}$  is defined as  $h_{\square}(n) := \inf\{\langle m, n \rangle \mid m \in \square\}$  and the polytope  $\square$  can be reconstructed from the support function

$$\square_h = \{m \in M \mid \langle m, n \rangle \geq h(n) \quad \forall n \in N\}. \quad (1)$$

The support function  $h_{\square}$  is piecewise linear in the sense that  $N_{\mathbb{R}}$  is the union of a non-empty finite collection of strongly convex polyhedral cones in  $N_{\mathbb{R}}$  such that  $h_{\square}$  is linear on each cone. A fan is a collection  $\Delta$  of strongly convex polyhedral cones in  $N_{\mathbb{R}}$  such that every face of  $\sigma \in \Delta$  is contained in  $\Delta$  and  $\sigma \cap \sigma' \in \Delta$  for all  $\sigma, \sigma' \in \Delta$ .

The *normal fan*  $\Delta$  is the coarsest fan such that  $h_{\square}$  is linear on each  $\sigma \in \Delta$ , i.e. for all  $\sigma \in \Delta$  there exists  $l_{\sigma} \in M$  such that

$$h_{\square}(n) = \langle l_{\sigma}, n \rangle \quad \forall n \in \sigma. \quad (2)$$

The 1-dimensional cones  $\rho \in \Delta$  are generated by unique primitive elements  $n(\rho) \in N \cap \rho$  such that  $\rho = \mathbb{R}_{\geq 0} n(\rho)$ .

Upon refinement of the normal fan, we can assume that two successive pairs of  $n(\rho)$ 's generate the lattice and we obtain the *refined normal fan*, which will be the fan we will be using for the the rest of the present paper.

The 2-dimensional *algebraic torus*  $T_N \simeq k^* \times k^*$  is defined by  $T_N := \text{Hom}_{\mathbb{Z}}(M, k^*)$ . The multiplicative character  $e(m)$ ,  $m \in M$  is the homomorphism  $e(m) : T \rightarrow k^*$  defined by  $e(m)(t) = t(m)$  for  $t \in T_N$ . Specifically, if  $\{n_1, n_2\}$  and  $\{m_1, m_2\}$  are dual  $\mathbb{Z}$ -bases of  $N$  and  $M$  and we denote  $u_j := e(m_j)$ ,  $j = 1, 2$ , then we have an isomorphism  $T_N \simeq k^* \times k^*$  sending  $t$  to  $(u_1(t), u_2(t))$ . For  $m = \lambda_1 m_1 + \lambda_2 m_2$  we have

$$e(m)(t) = u_1(t)^{\lambda_1} u_2(t)^{\lambda_2}. \quad (3)$$

The *toric surface*  $X_{\square}$  associated to the refined normal fan  $\Delta$  of  $\square$  is

$$X_{\square} = \bigcup_{\sigma \in \Delta} U_{\sigma}$$

where  $U_{\sigma}$  is the  $k$ -valued points of the affine scheme  $\text{Spec}(k[\mathcal{S}_{\sigma}])$ , i.e., morphisms  $u : \mathcal{S}_{\sigma} \rightarrow k$  with  $u(0) = 1$  and  $u(m + m') = u(m)u(m') \quad \forall m, m' \in \mathcal{S}_{\sigma}$ , where  $\mathcal{S}_{\sigma}$  is the additive subgroup of  $M$

$$\mathcal{S}_{\sigma} = \{m \in M \mid \langle m, y \rangle \geq 0 \quad \forall y \in \sigma\}.$$

The *toric surface*  $X_{\square}$  is irreducible, non-singular and complete under the assumption that we are working with the refined normal fan. If  $\sigma, \tau \in \Delta$  and  $\tau$  is a face of  $\sigma$ , then  $U_{\tau}$

is an open subset of  $U_\sigma$ . Obviously  $\mathcal{S}_0 = M$  and  $U_0 = T_N$  such that the algebraic torus  $T_N$  is an open subset of  $X_\square$ .

$T_N$  acts algebraically on  $X_\square$ . On  $u \in U_\sigma$  the action of  $t \in T_N$  is obtained as

$$(tu)(m) := t(m)u(m) \quad \text{for } m \in \mathcal{S}_\sigma,$$

such that  $tu \in U_\sigma$  and  $U_\sigma$  is  $T_N$ -stable. The orbits of this action are in one-to-one correspondence with  $\Delta$ . For each  $\sigma \in \Delta$ , let

$$\text{orb}(\sigma) := \{u : M \cap \sigma \rightarrow k^* | u \text{ is a group homomorphism}\}.$$

Then  $\text{orb}(\sigma)$  is a  $T_N$  orbit in  $X_\square$ . Define  $V(\sigma)$  to be the closure of  $\text{orb}(\sigma)$  in  $X_\square$ .

A  $\Delta$ -linear support function  $h$  gives rise to a polytope  $\square$  as above and an associated Cartier divisor

$$D_h = D_\square := - \sum_{\rho \in \Delta(1)} h(n(\rho)) V(\rho),$$

where  $\Delta(1)$  is the 1-dimensional cones in  $\Delta$ . In particular

$$D_m = \text{div}(\mathbf{e}(-m)) \quad m \in M.$$

**Lemma 1.** *Let  $h$  be a  $\Delta$ -linear support function with associated convex polytope  $\square$  and Cartier divisor  $D_h = D_\square$ . The vector space  $H^0(X, \mathcal{O}_X(D_h))$  of global sections of  $\mathcal{O}_X(D_\square)$ , i.e., rational functions  $f$  on  $X_\square$  such that  $\text{div}(f) + D_\square \geq 0$  has dimension  $\#(M \cap \square)$  and has  $\{\mathbf{e}(m) | m \in M \cap \square\}$  as a basis.*

### B. Intersection theory on a toric surface

For a  $\Delta$ -linear support function  $h$  and a 1-dimensional cone  $\rho \in \Delta(1)$  we will determine the intersection number  $(D_h; V(\rho))$  between the Cartier divisor  $D_h$  and  $V(\rho) = \mathbb{P}^1$ . This number is obtained in [9, Lemma 2.11]. The cone  $\rho$  is the common face of two 2-dimensional cones  $\sigma', \sigma'' \in \Delta(2)$ . Choose primitive elements  $n', n'' \in N$  such that

$$\begin{aligned} n' + n'' &\in \mathbb{R}\rho \\ \sigma' + \mathbb{R}\rho &= \mathbb{R}_{\geq 0}n' + \mathbb{R}\rho \\ \sigma'' + \mathbb{R}\rho &= \mathbb{R}_{\geq 0}n'' + \mathbb{R}\rho \end{aligned}$$

**Lemma 2.** *For any  $l_\rho \in M$ , such that  $h$  coincides with  $l_\rho$  on  $\rho$ , let  $\bar{h} = h - l_\rho$ . Then*

$$(D_h; V(\rho)) = -(\bar{h}(n') + \bar{h}(n'')).$$

In the 2-dimensional non-singular case let  $n(\rho)$  be a primitive generator for the 1-dimensional cone  $\rho$ . There exists an integer  $a$  such that

$$n' + n'' + an(\rho) = 0,$$

$V(\rho)$  is itself a Cartier divisor and the above gives the self-intersection number

$$(V(\rho); V(\rho)) = a.$$

More generally the self-intersection number of a Cartier divisor  $D_h$  is obtained in [9, Prop. 2.10].

**Lemma 3.** *Let  $D_h$  be a Cartier divisor and let  $\square_h$  be the polytope associated to  $h$ . Then*

$$(D_h; D_h) = 2 \text{vol}_2(\square_h),$$

where  $\text{vol}_2$  is the normalized Lebesgue-measure.

### C. The support of the codes

The toric codes are obtained from evaluating certain rational functions in a suitable set  $S$  of  $\mathbb{F}_q$ -rational points on toric varieties, being the intersection of two ample divisors on  $X$ .

**Definition 4.** For  $i = 1, 2$  let  $I_i, J_i \subseteq \mathbb{F}_q^*$  with  $I_1 \cap J_2 = I_2 \cap J_1 = \emptyset$  and introduce the two rational functions

$$F_i = \prod_{\psi \in I_i} (e(m_1) - \psi)^{n_{1,\psi}} \prod_{\psi \in J_i} (e(m_2) - \psi)^{n_{2,\psi}},$$

where the integer exponents satisfy  $n_{1,\psi} \geq 1$  and  $n_{2,\psi} \geq 1$ .

For  $i = 1, 2$ , let  $D_i = (F_i)_0$  be their divisor of zeroes,  $|D_i|$  be their support and  $U_i = X \setminus |D_i|$  their complement. It is important to note that the supports and their complement are independent of the choice of the exponents  $n_{1,\psi} \geq 1$  and  $n_{2,\psi} \geq 1$ .

Finally let the support set of the code be  $S = |D_1| \cap |D_2| = U_1 \cup U_2 \subseteq \mathbb{F}_q^* \times \mathbb{F}_q^*$ .

**Remark 5.** As a set  $S = I_1 \times J_2 \cup I_2 \times J_1 \subseteq \mathbb{F}_q^* \times \mathbb{F}_q^*$  with  $\#S = \#I_1 \cdot \#J_2 + \#I_2 \cdot \#J_1$  elements, but it is important to have in mind, that  $S \subseteq \mathbb{F}_q^* \times \mathbb{F}_q^*$  is realized as the support of the intersection of two divisors in many different ways, namely one for each choice of the exponents  $n_{1,\psi} \geq 1$  and  $n_{2,\psi} \geq 1$ .

### D. Toric evaluation codes

We start by exhibiting the toric codes as evaluation codes supported on  $S$ .

**Definition 6.** For each  $t \in T \simeq k^* \times k^*$ , we evaluate the rational functions in  $H^0(X, \mathcal{O}_X(D_\square))$

$$\begin{aligned} H^0(X, \mathcal{O}_X(D_\square)) &\rightarrow k \\ f &\mapsto f(t). \end{aligned}$$

Let  $H^0(X, \mathcal{O}_X(D_\square))^{\text{Frob}}$  denote the rational functions in  $H^0(X, \mathcal{O}_X(D_\square))$  that are invariant under the action of the Frobenius, that is functions that are  $\mathbb{F}_q$ -linear combinations of the functions  $\mathbf{e}(m)$  in (3).

Evaluating in all points in  $S$ , we obtain the code  $C_{S,\square} \subset (\mathbb{F}_q)^{\#S}$  as the image

$$\begin{aligned} H^0(X, \mathcal{O}_X(D_h))^{\text{Frob}} &\rightarrow C_{S,\square} \subset (\mathbb{F}_q)^{\#S} \\ f &\mapsto (f(t))_{t \in T(\mathbb{F}_q)} \end{aligned}$$

and the generators of the code is obtained as the image of the basis

$$\mathbf{e}(m) \mapsto (\mathbf{e}(m)(t))_{t \in S}.$$

as in (3).

To estimate the parameters we have two bound the number of points in the support  $S$ , where the rational functions in  $H^0(X, \mathcal{O}_X(D_\square))^{\text{Frob}}$  evaluates to zero.

The support  $S$  is stratified by the intersections with the zeros of  $\mathbf{e}(m_1) - \psi$ , where  $\psi \in I_1 \cup I_2 \subseteq \mathbb{F}_q^*$ . A rational function  $f$  can either vanish identically on a stratum or have a finite number of zeroes along the stratum.

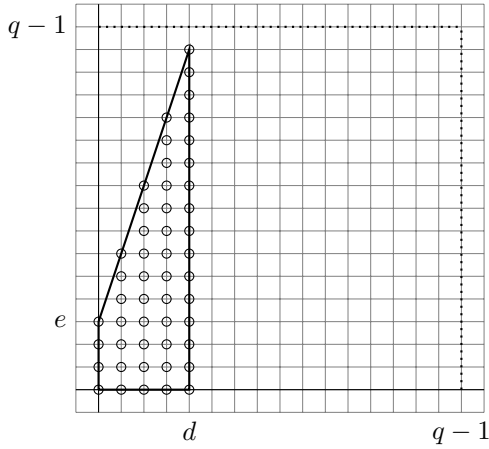


Fig. 1. The polytope of Theorem 8 is the polytope with vertices  $(0, 0)$ ,  $(d, 0)$ ,  $(d, e + rd)$ ,  $(0, e)$ .

	$V(\rho_1)$	$V(\rho_2)$	$V(\rho_3)$	$V(\rho_4)$
$V(\rho_1)$	$-r$	1	0	1
$V(\rho_2)$	1	0	1	0
$V(\rho_3)$	0	1	$r$	1
$V(\rho_4)$	1	0	1	0

TABLE I

THE INTERSECTION NUMBERS FOR THE FOUR 1-DIMENSIONAL CONES OF THE FAN OF THE HIRZEBRUCH SURFACE

1) *Identically vanishing*: Assume that  $f$  is identically zero along precisely  $a$  of these strata. As  $\mathbf{e}(m_1) - \psi$  and  $\mathbf{e}(m_1)$  have the same divisors of poles, they have equivalent divisors of zeroes, so

$$(\mathbf{e}(m_1) - \psi)_0 \sim (\mathbf{e}(m_1))_0.$$

Therefore

$$\text{div}(f) + D_{\square} - a(\mathbf{e}(m_1))_0 \geq 0$$

or equivalently

$$f \in H^0(X, \mathcal{O}_X(D_{\square} - a(\mathbf{e}(m_1))_0)).$$

Depending on the polytope  $\square$  this gives an upper bound for the number  $a$ , using Lemma 1.

2) *Vanishing in a finite number of points*: On any of the  $\#I_1 \cup \#I_2 - a$  other strata the number of zeroes of  $f$  is according to [10] at most the intersection number

$$(D_{\square} - a(\mathbf{e}(m_1))_0; (\mathbf{e}(m_1))_0). \quad (4)$$

This number can be calculated using Lemma 2 and Lemma 3.

The above gives a method to construct toric codes from surfaces and obtain their precise parameters, this was done by the author in four cases in [2].

**Example 7.** (Hirzebruch surfaces). Let  $d, e, r$  be positive integers and let  $\square$  be the polytope in  $M_{\mathbb{R}}$  with vertices  $(0, 0)$ ,  $(d, 0)$ ,  $(d, e + rd)$ ,  $(0, e)$ , see Figure 1 and with (refined) normal fan as in Figure 2.

From the Hirzebruch surfaces with  $I_1 = J_2 = \mathbb{F}_q^* \times \mathbb{F}_q^*$  and  $I_2 = J_1 = \emptyset$ , we obtain using the above method the following theorem.

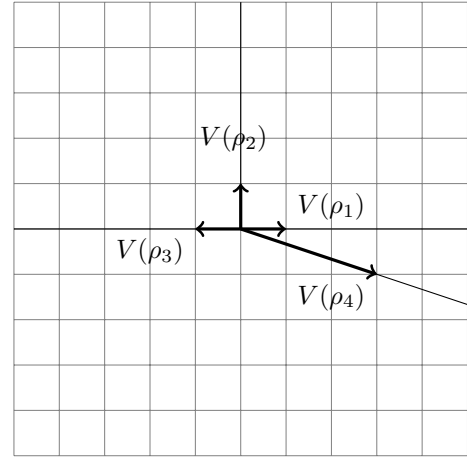


Fig. 2. The normal fan and the 1-dimensional cones of the polytope in Fig. 1 that gives rise to a Hirzebruch surface.

**Theorem 8.** Assume that  $d < q - 1$ , that  $e < q - 1$  and that  $e + rd < q - 1$ . The toric code  $C_{\mathbb{F}_q^* \times \mathbb{F}_q^*, \square}$  has length equal to  $(q - 1)^2$ , dimension equal to  $\#(M \cap \square) = (d + 1)(e + 1) + r \frac{d(d+1)}{2}$  (the number of lattice points in  $\square$ ) and the minimal distance is equal to  $\text{Min}\{(q - 1 - d)(q - 1 - e), (q - 1)(q - 1 - e - rd)\}$ .

D. Joyner [11] has done extensive calculations on among others these toric codes. R. Joshua and R. Akhtar [12] have obtained results on a different kind of toric codes that appear to be related to the dual of the present codes.

### III. CODES FROM TORIC SURFACES CONTAINING THEIR DUAL CODE

#### A. Differential forms and residues

The residue theorem is obtained in [13] over  $\mathbb{C}$ , however the theorem and its various forms are essential and for completeness we present general proofs here. Throughout  $\text{Res}_P(\omega)$  means the local Grothendieck residue see, e.g., [14] and [15]. For residues on toric varieties we also refer to [16].

**Theorem 9** (Residue theorem - general form). *Let  $X$  be a complete smooth algebraic surface and let  $\omega_X$  be the sheaf of differential 2-forms on  $X$ . Let  $U_1, U_2$  be two open subsets of  $X$  such that  $X \setminus (U_1 \cup U_2) = S$  is a finite set of points. Then*

- Let  $\omega \in \omega_X(U_1 \cup U_2) = H^0(U_1 \cup U_2, \omega_X)$  be any 2-form on  $X$  with no poles on  $U_1 \cup U_2$ , then  $\sum_{P \in S} \text{Res}_P(\omega) = 0$ .*
- For any  $(w_{\bullet}) \in \bigoplus_{P \in S} k$  with  $\sum_{P \in S} w_P = 0$ , there exists an  $\omega \in \omega_X(U_1 \cup U_2) = H^0(U_1 \cup U_2, \omega_X)$ , such that  $\text{Res}_P(\omega) = w_P$  for all  $P \in S$ .*

*Proof:* The Čech resolution  $\omega_X|_{U_1} \amalg \omega_X|_{U_2} \rightarrow \omega_X|_{U_1 \cup U_2}$  of the sheaf  $\omega_X|_{U_1 \cup U_2}$ , obtained from the two open sets  $U_1$  and  $U_2$ , gives that a 2-form  $\omega$  on  $X$  without poles on  $U_1 \cup U_2$  defines a class  $[\omega] \in H^1(U_1 \cup U_2, \omega_X)$  and that every class has such a representation.

As  $H^2(X, \omega_X) \cong k$  and  $H^i(U_1 \cup U_2, \omega_X) = 0$  for  $i \geq 2$  by

Serre duality, relative cohomology gives the exact sequence

$$\begin{array}{ccccc} H^1(U_1 \cup U_2, \omega_X) & \longrightarrow & H_S^2(X, \omega_X) & \longrightarrow & H^2(X, \omega_X) \\ \parallel & & \downarrow \cong & & \downarrow \cong \\ & & \bigoplus_{P \in S} H_P^2(X, \omega_X) & & \\ & & \downarrow \cong & & \\ H^1(U_1 \cup U_2, \omega_X) & \xrightarrow{\text{Res}_{[\omega] \mapsto \bigoplus_{P \in S} \text{Res}_P(\omega)}} & \bigoplus_{P \in S} k & \xrightarrow{\Sigma} & k \end{array}$$

Then  $\sum_{P \in S} \text{Res}_P(\omega) = \text{Res}([\omega])$  and the claims follows from exactness of the last sequence. ■

In the above form there is no restrictions on the polar behavior as long as there are no poles on  $U_1 \cup U_2$ , however it is possible to prove the theorem in a stronger form.

For a divisor  $D$  on  $X$  the sheaf of differential forms  $\omega_X(D)$ , is the sheaf with  $\omega_X(D)(U) = \{\eta \in \omega(U) | (\eta) + D \geq 0 \text{ on } U\}$  on open sets  $U \subseteq X$ . Its global sections  $H^0(X, \omega(D))$  are the differential forms  $\omega$  with  $(\omega) + D \geq 0$ .

**Theorem 10** (Residue theorem - special form). *Let  $X$  be a complete smooth algebraic surface and let  $\omega_X$  be the sheaf of differential 2-forms on  $X$ . For  $i = 1, 2$ , let  $D_i$  be ample and effective divisors on  $X$  with support  $|D_i|$  and with complement  $U_i = X \setminus |D_i|$ . Assume that  $X \setminus (U_1 \cup U_2) = |D_1| \cap |D_2| = S$  is a finite set of points.*

- i) *For any  $\omega \in H^0(X, \omega(D_1 + D_2))$ , we have that  $\sum_{P \in S} \text{Res}_P(\omega) = 0$ .*
- ii) *For any  $(w_\bullet) \in \bigoplus_{P \in S} k$  with  $\sum_{P \in S} w_P = 0$ , there exists an  $\omega \in H^0(X, \omega(D_1 + D_2))$ , such that  $\text{Res}_P(\omega) = w_P$  for all  $P \in S$ .*

*Proof:* The morphism of sheaves

$$\omega_X(D_1) \oplus \omega_X(D_2) \rightarrow \omega_X(D_1 + D_2)$$

is injective with cokernel  $j_*(\omega|_{U_1 \cap U_2})$ , where  $j$  is the open immersion of  $U_1 \cup U_2 \hookrightarrow X$ . The associated long exact cohomology sequence gives a surjection  $H^0(X, \omega(D_1 + D_2)) \rightarrow H^1(U_1 \cup U_2, \omega_X)$  as  $H^1(X, \omega(D_1)) = H^1(X, \omega(D_2)) = 0$  by the assumption on ampleness of the divisors.

The proof now follows as in the above proof of Theorem 9. ■

### B. Dualizing differential form of a toric code

We want to exhibit a differential form  $\omega_0$  on  $X$  with poles restricted to the points in the support  $S = |D_1| \cap |D_2| = U_1 \cup U_2 \subseteq \mathbb{F}_q^* \times \mathbb{F}_q^*$ , where  $D_i = (F_i)_0$  are divisors of zeroes of the functions defined in Definition 4 and  $|D_i|$  are their support and  $U_i = X \setminus |D_i|$  their complement. Besides that we want the differential form  $\omega$  to vanish at the divisor  $2D_\square$ .

**Definition 11.** A differential form  $\omega_0 \in H^0(X, \omega(D_1 + D_2 - 2D_\square))$  is called a dualizing form for the toric code and we will call the set

$$R = \{P \in S | \text{Res}_P(\omega_0) \neq 0\} \subseteq S$$

its restricted support.

This existence of a dualizing form for the toric code is obtained in two steps utilizing the representations of the set  $S$  as the intersection of the supports of various ample divisors.

**Theorem 12.** *Assume that the support of the toric code is the intersection of the support of two ample divisors as i Definition 4. Assume that we can choose large exponents  $n_{1,\psi} \geq 1$  and  $n_{2,\psi} \geq 1$ , such that  $L(D_1 + D_2 - 2D_\square) \neq \emptyset$ . Then there exists a dualizing form for the toric code of Definition 6 with ample divisors  $D_1$  and  $D_2$ .*

*Proof:* For  $i = 1, 2$ , let  $D_i = (F_i)_0$  be their divisor of zeroes,  $|D_i|$  be their support and  $U_i = X \setminus |D_i|$  their complement. Assuming that we can choose the exponents  $n_{1,\psi} \geq 1$  and  $n_{2,\psi} \geq 1$  such that  $D_i$  are ample Theorem 10 gives that for any  $(w_\bullet) \in \bigoplus_{P \in S} k$  with  $\sum_{P \in S} w_P = 0$ , there exists an  $\omega \in H^0(X, \omega(D_1 + D_2))$ , such that  $\text{Res}_P(\omega) = w_P$  for all  $P \in S$ .

In order to find a differential form vanishing at the divisor  $2D_\square$  we note that the support of the divisors  $D_1$  and  $D_2$  and their complement is independent of the choice of the exponents  $n_{1,\psi} \geq 1$  and  $n_{2,\psi} \geq 1$ , see Remark 5. An  $\omega$  constructed as above is in the corresponding  $H^0(X, \omega(D_1 + D_2))$  for larger values of the exponents and the corresponding divisors  $D_1$  and  $D_2$  are still ample.

Choose large exponents  $n_{1,\psi} \geq 1$  and  $n_{2,\psi} \geq 1$ , such that  $L(D_1 + D_2 - 2D_\square) \neq \emptyset$  and let  $F \neq 0$  in  $L(D_1 + D_2 - 2D_\square)$  using Lemma 1.

The corresponding divisors  $D_1$  and  $D_2$  and the differential form  $\omega_0 = F\omega \in H(X, \omega(D_1 + D_2 - 2D_\square))$  are the desired entities. ■

### C. Toric codes contained in their dual codes

For a linear code  $C \subseteq \mathbb{F}_q^n$  and  $w \in (\mathbb{F}_q^*)^n$ , we let the  $w$ -dual be the code

$$C_w^\perp = \{x \in \mathbb{F}_q^n | \sum_{i=1}^n w_i x_i y_i = 0 \ \forall y \in C\} \subseteq \mathbb{F}_q^n.$$

If  $w = (1, 1, \dots, 1)$  then  $C_w^\perp$  is the usual dual code  $C^\perp$ . In the notation above, we have the following theorem.

**Theorem 13.** *Let  $C_\square$  be a toric code. Assume that the support  $S$  of the toric code is the intersection of the support of two ample divisors as i Definition 4. Let  $\omega_0 \in H^0(X, \omega(D_1 + D_2 - 2D_\square))$  be a dualizing differential for the code with ample divisors  $D_1$  and  $D_2$  and let  $R \subseteq S$  be the corresponding restricted support of the code. The evaluation code  $C_{R,\square}$  obtained from evaluating functions in  $L(D_\square)$  at points in the restricted support  $R$  satisfies  $C_{R,\square} \subseteq (C_{R,\square})_{w_\bullet}^\perp$ , where  $w_P = \text{Res}(\omega_0)_P$  for  $P \in R$ .*

*Proof:* For any  $f, g \in L(D_\square)$  and  $\omega_0 \in H(X, \omega(D_1 + D_2 - 2D_\square))$ , we have that  $fg\omega_0 \in H(X, \omega(D_1 + D_2))$ . By Theorem 10  $\sum_{P \in R} f(P)g(P) \text{Res}(\omega_0)_P = 0$ , which proves the claim. ■

If the weights  $w_i$  are squares  $w_i = v_i^2$ , which is always the case if  $q = 2^m$ , let  $g_v$  be coordinatwise multiplication by  $v_\bullet$ . Then the code  $\tilde{C} = g_v(C)$  satisfy  $\tilde{C} \subseteq \tilde{C}^\perp$  and the dual code  $C' = \tilde{C}^\perp$  satisfy  $C'^\perp \subseteq C'$ .

This code can be used in the Calderbank-Shor-Steane method for constructing quantum error correcting codes.

**Example 14.** (Hirzebruch Surfaces) The toric codes are obtained from evaluating certain rational functions in a suitable

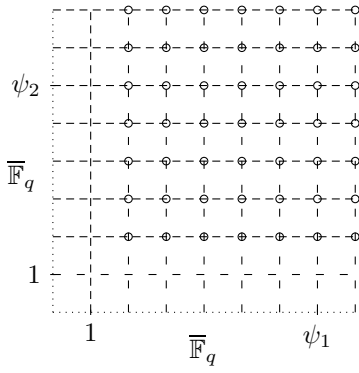


Fig. 3. Illustration of the supports  $|D_1|$  and  $|D_2|$  of the divisors  $D_1$  (loosely dashed) and  $D_2$  (densely dashed) and the support of the code  $S = |D_1| \cap |D_2|$  having  $(q-2)^2$  points.

set  $S$  of rational points on toric varieties, being the intersection of two ample divisors on  $X$  as in Definition 4. With

$$F_1 = \prod_{\psi \in \mathbb{F}_q^* \setminus \{1\}} (e(m_1) - \psi) (e(m_2) - 1)$$

$$F_2 = (e(m_1) - 1) \prod_{\psi \in \mathbb{F}_q^* \setminus \{1\}} (e(m_2) - \psi)$$

we have the divisors

$$D_1 = (F_1)_0 \sim (q-2)(V(\rho_1) + rV(\rho_4)) + V(\rho_2)$$

$$D_2 = (F_2)_0 \sim (V(\rho_1) + rV(\rho_4)) + (q-2)V(\rho_2)$$

as

$$(\mathbf{e}(m_1) - \psi)_0 \sim (\mathbf{e}(m_1))_0 \sim V(\rho_1) + rV(\rho_4)$$

$$(e(m_2) - \psi)_0 \sim V(\rho_2) .$$

The divisors  $D_1$  and  $D_2$  are seen to be ample on  $X$ , using the intersection numbers in Table 7 and the Nakai criterion. The support set of the code  $S = |D_1| \cap |D_2| = U_1 \cup U_2 = (\mathbb{F}_q^* \setminus \{1\}) \times (\mathbb{F}_q^* \setminus \{1\}) \subseteq \mathbb{F}_q^* \times \mathbb{F}_q^*$  is realized as the intersection of the support of two ample divisors and we can apply the construction above.

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